SUBJECTIVE GEOMETRY.

BY DR. G. W. HILL.

(Read before the American Mathematical Society, February 29, 1908.)

THE fact that many persons suppose that the non-euclidean and euclidean geometries contradict each other seems to call for further elucidation of the matter. Things that disagree do not necessarily contradict each other. One circumstance that leads to the prevalent obscurity in this matter is the use of diagrams and objective phraseology in the treatment of synthetic geometry. The reader always supposes that geometric figures are something he can see and handle. Also, in Euclid's mode of presenting the science, it is difficult to see what is fundamental and what derivative.

It is said that Legendre cast about for a long time to find the proof of the theorem "The sum of the three angles of a triangle is equivalent to two right angles" without success. He should have seen that his failure arose from the want of due determination in his definition. The theorem is true if the proper concepts are attached to the words "triangle," "angle" and "right angle."

Let us illustrate by an instance from algebra. The man who knows nothing about negative and complex quantity will say "No cubic equation can have more than three roots, but there are some which have less." The man who admits these sorts of quantity says "Every cubic equation has three roots." These statements disagree, but are not in contradiction. The case is the same as if they were written in different languages. We can translate the first statement so as to agree with the second by simply writing "positive real root" instead of "root."

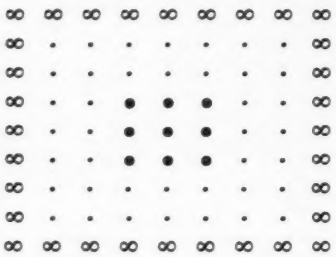
The difference between the non-euclidean and the euclidean geometry is one of mere definition of terms. We shall see this more easily if we treat geometry as a subjective science. There are two methods for its exposition called synthetic and analytic. The latter makes use of the symbolic notation of algebra, and is more easily comprehended if the notion of a linear unit is admitted. At the end of the investigation, however, the latter

can be brushed away as mere scaffolding, and what is left is magnitudinal geometry in place of mensuration.

We shall employ the analytic method. First it is necessary to understand what is meant by a manifold. Let x be a variable capable of taking all real values, and let these values be arranged in the order of their relative magnitudes. This is the manifold of one dimension. We may objectively illustrate it by the graph

$$\infty, \cdots -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5 \cdots, \infty,$$

exhibiting the two series of positive and negative integers separated by 0. The symbol ∞ here simply denotes unlimited extension. The values x may take are called the elements of the manifold. Again we may suppose another variable y, independent of x but able to take a series of values in the same way. Now let this manifold of one dimension exist in conjunction with each one of the elements of the manifold of one dimension for x; that is, there is constituted a sort of table of double entry. This is the manifold of two dimensions in reference to the two variables x and y. The element of the second manifold is formed by joining any value of x to any value of y. The manifold of two dimensions may be illustrated thus (the square of heavy dots denotes contemplated elements, the light dots indicated elements, and the whole is bordered with the symbol ∞):



Proceeding in like manner, and adopting a third variable z, we can form the manifold of three dimensions; and so on.

Let us use the letter M for manifold. Thus the three manifolds we have considered are designated by the symbols

$$M(x), \qquad M(x, y), \qquad M(x, y, z).$$

Now let us suppose that a, b, c denote three constant quantities, and let the first be added to every value of x, the second to every value of y, and the third to every value of z. Then it is evident we may write the identities

$$M(x + a) \equiv M(x), \quad M(x + a, y + b) \equiv M(x, y),$$
$$M(x + a, y + b, z + c) \equiv M(x, y, z).$$

The property of a manifold shown by these identities may be called the isotropy of transposition. It is important to bear in mind that we have established this property only by insisting that the boundary of the manifold is indefinitely removed.

We must here explain some metaphysical terms. The attributes of an entity are whatever may be rightfully considered as belonging or attached to it. We divide attributes into the two classes of properties and characteristics. Properties are attributes which cannot be ignored, they are essential to the constitution of the entity. But characteristics admit of choice. To apply this: a manifold has but one property, viz., that of the isotropy of transposition.

We can now consider the connection of the manifold of any number of dimensions with geometry of the same number of dimensions. Geometry is the science of space. What is space? It is the continuum which holds the manifold. Is anything contained in the continuum besides the manifold? If it is subjective geometry we are dealing with, the answer is nothing. Then the attributes of space are identical with those of the manifold. In our reasonings we may everywhere substitute the word manifold for the word space.

To make an application of what we have discovered: It has been questioned whether Euclid was warranted in employing his method of proof by superposition. There need not be any question about this, for the isotropy of transposition justifies it.

We proceed now to investigate possible characteristics for the manifold. Since the manifold is unique for a given number of dimensions, it follows that the number of independent characteristics, necessary for thoroughly defining the manifold, is equal to the number of its dimensions. We can build up the system of characteristics by starting from the manifold of one dimension. In a manifold of n dimensions, if we suppose one of the variables to be limited to a single value, and, as another step, the variable to be disregarded, the result

is a manifold of $n-1$ dimensions. Then, in selecting the characteristics for our manifold, we can suppose the characteristics of the manifold of two dimensions are congruent with those of the manifold of one dimension. And, again, that the characteristics of the manifold of three dimensions are congruent with those of two dimensions, and so on. For the sole function of geometry is to ascertain the relativities of special elements of the manifold. We therefore start with the manifold of one dimension.

Let the special values of the variable x be denoted by x' and x''. Here the only possible relativities are two, viz., $x''-x'$ and $x'-x''$. But we insist that these relativities shall be divided so that one is always positive and the other shall express the idea of advance or recession. Then, calling the first distance, and denoting it by D, we might write the equation $D=\pm(x''-x')$. But, not liking the ambiguous sign, we prefer to write

$$D=\sqrt{(x''-x')^2},$$

where the radical must be taken positively. Then, similarly, for the manifold of one dimension, whose variable is y, we have

$$D=\sqrt{(y''-y')^2},$$

and, again, for one whose variable is z,

$$D=\sqrt{(z''-z')^2}.$$

How can D, for two dimensions, be made congruent with the D of one dimension? And, again, the D of three dimensions congruent with the D of two dimensions? It is plain that this will be accomplished in the simplest manner, if, in the first case, we put

$$D=\sqrt{(x''-x')^2+(y''-y')^2},$$

and, in the second case,

$$D=\sqrt{(x''-x')^2+(y''-y')^2+(z''-z')^2}.$$

For higher dimensions we shall have similar expressions. It must be borne in mind that these things are matters for choice.

What we have done suffices for the manifold of one dimension. But, for the manifold of two dimensions, we must have an additional characteristic. We call this λ, and choose to put

$$\frac{x''-x'}{D}=F_1(\lambda),\quad \frac{y''-y'}{D}=F_2(\lambda),$$

where F_1 and F_2 denote two functions not yet determined. We derive from these equations

$$[F_1(\lambda)]^2+[F_2(\lambda)]^2=1,$$

and

$$F_1(\lambda)=\pm\sqrt{1-[F_2(\lambda)]^2}.$$

Thus it is plain that the corresponding values of $F_1(\lambda)$ and $F_2(\lambda)$ can be tabulated in a table to single entry, that F_1 and F_2 are both periodic functions of λ to the same, but arbitrary, period, and that the phase of F_2 is a quarter of a period behind the phase of F_1. Consequently, we may put

$$F_1(\lambda)=\cos(m\lambda+\alpha),\quad F_2(\lambda)=\sin(m\lambda+\alpha),$$

where m and α are arbitrary constants. The latter are at our disposal, only m must not, for efficiency's sake, receive either of the values $0, \infty$. Thus, for the sake of simplicity, we put $m=1$, $\alpha=0$. This comes to the same thing as saying that, what we first called $m\lambda+\alpha$, we now designate as λ. It must be remembered that we have put no restrictions on λ, it simply has the capacity of variation. Now we have

$$F_1(\lambda)=\cos\lambda,\quad F_2(\lambda)=\sin\lambda,$$

and the period of the two functions is 2π.

In the case of three dimensions, adopting a third variable θ, we choose to put

$$\frac{x''-x'}{D}=F_1(\lambda,\theta),\quad \frac{y''-y'}{D}=F_2(\lambda,\theta),\quad \frac{z''-z'}{D}=F_3(\lambda,\theta),$$

whence is derived

$$[F_1(\lambda,\theta)]^2+[F_2(\lambda,\theta)]^2+[F_3(\lambda,\theta)]^2=1.$$

In attempting the solution of this functional equation, we note that it is not necessary for our purposes that we should arrive at a general solution, but only at any particular solution that is efficient. Thus we readily see that

$$F_1(\lambda, \theta) = \cos(m'\theta + \beta)\cos(m\lambda + \alpha),$$
$$F_2(\lambda, \theta) = \cos(m'\theta + \beta)\sin(m\lambda + \alpha),$$
$$F_3(\lambda, \theta) = \sin(m'\theta + \beta),$$

where m, m', α, β are arbitrary constants, which may have values adopted at will, except that m and m' must not receive either of the values 0, ∞. For like reasons as before, we assume $m = m' = 1$, $\alpha = \beta = 0$. Thus we have

$$\frac{x'' - x'}{D} = \cos\theta\cos\lambda, \quad \frac{y'' - y'}{D} = \cos\theta\sin\lambda, \quad \frac{z'' - z'}{D} = \sin\theta.$$

These formulas are purposely made to be congruent with those for two dimensions; if we make $\theta = 0$, and then disregard it and z, the formulas reduce to those for two dimensions.

It is not deemed necessary to notice the formulas for higher dimensions than three; they can be found in Jacobi, Vorlesungen über Dynamik, page 185.

We must notice the limits between which the new variables D, λ, θ play. While the old variables x, y, z play each between $-\infty$ and $+\infty$, D plays between 0 and $+\infty$, λ between 0 and 2π and θ between $-\pi/2$ and $+\pi/2$. It is not forbidden to extend λ and θ beyond these limits, but it must be borne in mind that the consequence of doing this is to repeat the field.

After this exposition it is easy to point out what distinguishes non-euclidean from euclidean geometry. In the first place the non-euclidean geometer insists that there may be something else in the continuum, which space is, beside the manifold; but does not define what it is. Of course it is = objective geometry — subjective geometry. In the second place he conceives that this vague something takes away from him the liberty of putting $m = 1$ in the argument $m\lambda + \alpha$. Accordingly he carries through all his ratiocinations in geometry allowing m to stand indeterminate. This course introduces no error into his science any more than the introduction of $i = \sqrt{-1}$ into arithmetic and algebra made those sciences erroneous. But it ought to be recognized as an innovation, to be justified by the advantages gained. With regard to $\sqrt{-1}$ it is almost the universal opinion of mathematicians that it is an improvement. But the approval of non-euclidean geometry is not so general; this may be due to its not having been

so long before the public. The gist of the argument in favor of this view of the science is that it is not necessary to fix the period of the variable expressing orientation, that is the value of m. But both classes of geometers put $\alpha = 0$; why should it be prohibited to put $m = 1$?*

As further illustrating the subject, let us consider the theorem Legendre wished to establish. We must begin by a definition. A triangle is the bundle of relativities of three elements of the manifold of two dimensions. Let these elements be denoted as

$$x', y' \quad \text{and} \quad x'', y' \quad \text{and} \quad x''', y'''.$$

Then we have the equations

$$(x'' - x') + (x''' - x'') + (x' - x''') = 0,$$
$$(y'' - y') + (y''' - y'') + (y' - y''') = 0.$$

We may adopt D, λ and D', λ' and D'', λ'', such that

$$x'' - x' = D\cos\lambda, \quad x''' - x'' = D'\cos\lambda', \quad x' - x''' = D''\cos\lambda'',$$
$$y'' - y' = D\sin\lambda, \quad y''' - y'' = D'\sin\lambda', \quad y' - y''' = D''\sin\lambda'',$$

where the λ are taken between the limits 0 and 2π. Consequently

$$D\cos\lambda + D'\cos\lambda' + D''\cos\lambda'' = 0,$$
$$D\sin\lambda + D'\sin\lambda' + D''\sin\lambda'' = 0.$$

From these equations may be derived the value of each of the distances expressed in terms of the others and the λ's. The equations determining these values can be written

$$D^2 = D''^2 - 2D''D'\cos[\pm\pi \pm (\lambda'' - \lambda')] + D'^2,$$
$$D'^2 = D^2 - 2DD''\cos[\pm\pi \pm (\lambda - \lambda'')] + D''^2,$$
$$D''^2 = D'^2 - 2D'D\cos[\pm\pi \pm (\lambda' - \lambda)] + D^2.$$

The ambiguous signs in the arguments of the cosines are to be so taken that each argument may result between the limits 0 and π. As there are six ambiguous signs, there are no less than 64 different combinations to be selected from. By a few simple considerations we get rid of the superfluous ones. We

* It is a curious psychological commentary on this matter that the astronomer, who has a personal equation in measuring angles of position, does unconsciously what both classes of geometers consciously refrain from doing.

can denote the arguments by A, B, C, A belonging to D, B to D' and C to D''. Thence the simpler forms

$$D^2 = D''^2 - 2D''D' \cos A + D'^2,$$
$$D'^2 = D^2 - 2DD'' \cos B + D''^2,$$
$$D''^2 = D'^2 - 2D'D \cos C + D^2.$$

A, B, and C are the angles of the triangle. From the circumstance that in writing the three arguments we have used a cyclical permutation of the symbols λ, λ', λ'', it follows that we are not permitted to choose at random either of the components of the second ambiguous sign, but if we take the upper in any one case we must so take it in all, and similarly with respect to the lower. This is no more than saying that $\lambda'' - \lambda'$ belongs exclusively to D, $\lambda - \lambda''$ to D' and $\lambda' - \lambda$ to D''. Thus we may write the two cases for the expressions of the arguments

$$\pm\pi + (\lambda'' - \lambda'),\ \pm\pi + (\lambda - \lambda''),\ \pm\pi + (\lambda' - \lambda),$$

and

$$\pm\pi - (\lambda'' - \lambda'),\ \pm\pi - (\lambda - \lambda''),\ \pm\pi - (\lambda' - \lambda).$$

We may now inquire what is the sum of the three arguments or angles. Thus

$$A + B + C = \pm\pi \pm\pi \pm\pi.$$

The condition that each angle is contained between the limits 0 and π makes the sum either π or 3π. But, in order that the latter value may hold, it is necessary that $A = B = C = \pi$; this is inadmissable for the reason that some triangles exist in which one of the angles is less than π, and this condition of things must be continuous with the former state. Thus the sum of the angles would move from 3π towards π which is contrary to the equation. The value 3π must be rejected, and we have

$$A + B + C = \pi,$$

which is what Legendre was desirous of proving.

Euclid's eleventh axiom is a clumsy way of rectifying the inadequate definition of what he meant by an angle; it is clearly introduced as an afterthought. This led Legendre and many others to think that this axiom was superfluous and to attempt to construct geometries in which it was not employed. If an

angle is described as a something that is attached to the vertex of a triangle, although it is shown how angles may be added, multiplied or divided, this proves nothing as to the actual magnitude of angles. The angles of a spherical triangle are also attached to the vertices and can be added, multiplied or divided in a similar way, but their actual magnitudes are different from the similar angles in a space of two dimensions. Attention ought to be directed to the fact that saying the angle is included between the sides of the triangle which come together and form the vertex adds nothing to the determination of the term "angle."

As a last point in criticism, many authors on geometry introduce misconceptions and misapprehensions into the matter by employing the concepts of the next higher dimension in a discussion which ought to be exclusively in terms belonging to the dimension dealt with. The student should be warned that lines, either straight or curved, have no existence in a space of one dimension, and that planes and curved surfaces do not exist in a space of two dimensions. This would put an end to the talk about a geometry of two dimensions on the surface of a sphere.

ON HIGHER CONGRUENCES AND MODULAR INVARIANTS.

BY PROFESSOR L. E. DICKSON.

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1. THE object of this paper is to give a two-fold generalization of Hurwitz's* explicit formula for the number of integral roots of a given congruence modulo p, p being prime. On the one hand, we may derive an equally simple formula which gives, apart from a multiple of p, the number of the roots of a specified order ($\leqq t$) of irrationality; viz., the roots belonging to the Galois field of order p^t. On the other hand, the problem may, without loss of simplicity, be further generalized †

* *Archiv der Math. u. Physik* (3), vol. 5 (1903), p. 17.

† Other generalizations of theoretical importance, but not leading readily to explicit expressions in terms of the coefficients, are given by H. Kühne, *Archiv der Math. u. Physik*, vol. 6 (1904), p. 174.

by replacing the initial field of integers modulo p by an arbitrary Galois field, $GF[p^n]$.

From his formula Hurwitz deduces an (absolute) invariant* of degree $p-1$ of the general binary form under linear transformations taken modulo p. He states that one of the fundamental questions in the theory of modular invariants is "die Frage der Endlichkeit": whether or not all the invariants can be expressed as rational integral functions with integral coefficients of a finite number of the invariants. Emphasizing the difficulty of this question, he answers it only for a special case. As a matter of fact, this question is trivial for modular invariants, since any polynomial in $a_1, \cdots, a_r$ with integral coefficients is congruent modulo p to a polynomial in which the exponent of each a_i is at most $p-1$ (in view of Fermat's theorem $a^p \equiv a$).

2. Consider an equation, with coefficients in any given Galois field $GF[p^n]$ of order p^n,

$$f(x) \equiv a_0 + a_1 x + \cdots + a_r x^r \equiv 0. \tag{1}$$

Let N denote the number of its roots, different from zero, which belong to the $GF[p^{nm}]$. For brevity, set $P = p^{nm}$. Then N is the number of the vanishing expressions $f(\xi)$, where ξ ranges over the $P-1$ marks $\neq 0$ of the $GF[P]$. In that field,

$$1 - k^{P-1} = \begin{cases} 1 \text{ if } k = 0, \\ 0 \text{ if } k \neq 0. \end{cases}$$

Hence $N \equiv N^* \pmod{p}$, where

$$N^* = \sum_{\xi} \{1 - [f(\xi)]^{P-1}\} \equiv -1 - \sum_{\xi} [f(\xi)]^{P-1}.$$

Employing the algebraic expansion

$$[f(\xi)]^{P-1} = \sum_{\mu} C_\mu \xi^\mu, \tag{2}$$

and summing for the marks $\xi \neq 0$ of the $GF[P]$, we get

$$N^* = -1 - \sum_{\mu} C_\mu \Big(\sum_{\xi} \xi^\mu\Big).$$

* For a cubic form this invariant is exhibited in expanded form for $p \leqq 13$ on p. 221 of my paper on modular invariants, *Trans. Amer. Math. Soc.*, vol. 8 (1907), pp. 205–230. At the time of writing that paper I did not know of Hurwitz's work.

By the writer's Linear Groups, page 54,

$$\sum_{\xi} \xi^{\mu} = \begin{cases} -1 \text{ for } \mu \text{ divisible by } P-1, \\ 0 \text{ for } \mu \text{ not divisible by } P-1. \end{cases}$$

Hence

$$N^* + 1 = \sum C_{\nu(P-1)} \qquad (\nu = 0, 1, \cdots). \tag{3}$$

Inserting the values of P and the C's, we get

$$N^* + 1 = \sum_{\alpha} \frac{(p^{nm}-1)!}{\alpha_0!\ \alpha_1! \cdots \alpha_r!} a_0^{\alpha_0} a_1^{\alpha_1} \cdots a_r^{\alpha_r}, \tag{4}$$

summed for all sets of integers $\alpha_i \geqq 0$ for which

$$\alpha_0 + \alpha_1 + \cdots + \alpha_r = p^{nm} - 1, \tag{5}$$

$$\alpha_1 + 2\alpha_2 + \cdots + r\alpha_r \equiv 0 \pmod{p^{nm}-1}. \tag{6}$$

In (4) the fractional form of the multinomial coefficient is to be replaced by its integral value. In fact, certain of the α's may be multiples of p *.

3. When $nm > 1$, certain of the multinomial coefficients in (4) are multiples of p and the corresponding terms may be dropped from the equation. To this end, set

$$\alpha_i = \sum_{j=0}^{mn-1} c_{ij} p^j \qquad (0 \leqq c_{ij} < p), \tag{7}$$

for $i = 0, 1, \cdots, r$. Further,

$$p^{nm} - 1 = \sum_{j=0}^{mn-1} (p-1) p^j.$$

Then, by well-known theorems, the multinomial coefficient M in (4) is a multiple of p unless

$$\sum_{i=0}^{r} c_{ij} = p-1 \quad (j = 0, 1, \cdots, mn-1); \tag{5'}$$

while, if these relations hold,

$$M \equiv \prod_{j=0}^{mn-1} \frac{(p-1)!}{c_{0j}!\, c_{1j}! \cdots c_{rj}!} \pmod{p}. \tag{8}$$

In place of (5) we may employ the more exacting relations (5').

* As this is not true in Hurwitz's case $n = m = 1$, we may then place the factor $(p-1)!$ before the summation sign.

Finally, the exponents of each a_i may be made less than p^n by employing

$$a_i^{p^n} = a_i. \tag{9}$$

4. Instead of deleting vanishing terms of (4) by employing relations (5′) in place of (5), we may modify our former method of determining the C_μ. Instead of the algebraic expansion (2) we may employ the modular relation

$$[f(\xi)]^{p^{nm}-1} \equiv \prod_{j=0}^{m-1} [f(\xi)]^{p^{nj}(p^n-1)} = \prod_{j=0}^{m-1} [f(\xi^{p^{nj}})]^{p^n-1}, \tag{10}$$

which follows readily from (9).

The plan of § 3, which employs (9) only at the final stage, has the advantage of furnishing simultaneously the number of roots in the $GF[p^t]$ of equation (1) with coefficients in any $GF[p^n]$, where n is a divisor of t. See the example in § 6.

5. In (1) set $x = x_2/x_1$ and consider the binary form

$$f(x_1, x_2) = a_0 x_1^r + a_1 x_1^{r-1} x_2 + \cdots + a_r x_2^r \tag{11}$$

in the $GF[p^n]$. Two pairs of marks (x_1, x_2) and (x_1', x_2') of the $GF[p^{nm}]$ will be said to give the same set of solutions of

$$f(x_1, x_2) = 0 \tag{12}$$

if, and only if, there exists a mark ρ of the $GF[p^{nm}]$ for which $x_1' = \rho x_1$, $x_2' = \rho x_2$ in that field. Let A denote the number of distinct sets of solutions, not both zero, of (12) in the $GF[p^{nm}]$. Evidently

$A = N$, when $a_0 \neq 0$ and $a_r \neq 0$ in the $GF[p^n]$;

$A = N + 1$, when just one of the coefficients a_0, a_r is zero;

$A = N + 2$, when $a_0 = a_r = 0$.

Hence in every case $A \equiv A^* \pmod p$, where

$$A^* + a_0^{p^n-1} + a_r^{p^n-1} = N + 2.$$

THEOREM. *The number A of distinct sets of solutions, not both zero, in the $GF[p^{nm}]$ of a homogeneous equation (12), with coefficients in the $GF[p^n]$, is congruent to A^* modulo p, where*

$$A^* - 1 = -a_0^{p^n-1} - a_r^{p^n-1} + \sum_{\alpha} \frac{(p^{nm}-1)!}{\alpha_0!\,\alpha_1!\cdots\alpha_r!} a_0^{\alpha_0} a_1^{\alpha_1} \cdots a_r^{\alpha_r}, \tag{13}$$

subject to (5) and (6), or to (6), (7) and (5′).

The function defined by the second member of (13) *is an absolute modular invariant of the binary form* (11).

6. Examples. Let $r = 3$, so that (11) is a binary cubic. First, let $p = 3$, $m = n = 1$; the invariant* is seen to be

$$K = a_1^2 + a_2^2 - a_0a_2 - a_1a_3. \tag{14}$$

Next, let $p = 3$, $nm = 2$. As in (7), set

$$\alpha_i = c_{i0} + 3c_{i1} \qquad (c_{ij} = 0, 1, 2).$$

Then, by (5′) and (6),

$$\sum_{i=0}^{3} c_{i0} = 2, \ \sum_{i=0}^{3} c_{i1} = 2, \ c_{10} + 2c_{20} + 3c_{30} + 3c_{11} + 6c_{21} + 9c_{31} \equiv 0 \pmod 8.$$

The sets of values of the c_{ij} are as follows:

c_{01}	c_{11}	c_{21}	c_{31}
2	0	0	0
1	0	1	0
1	1	0	0
1	0	1	0
0	0	1	1
0	2	0	0
0	0	2	0
0	1	0	1
0	0	2	0
0	0	0	2
0	1	0	1
0	2	0	0

c_{00}	c_{10}	c_{20}	c_{30}
2	0	0	0
1	0	1	0
0	0	1	1
0	2	0	0
1	1	0	0
1	0	1	0
0	0	2	0
0	0	2	0
0	1	0	1
0	0	0	2
0	1	0	1
0	2	0	0

In view of (8), the resulting function (13) is

$$\begin{aligned} &a_0^4a_2^4 + a_0^3a_1^3a_2a_3 - a_0^3a_1^2a_2^3 + a_0a_1a_2^3a_3^3 - a_0a_1^6a_2 \\ &\quad + a_2^8 - a_1^3a_2^2a_3^3 - a_1a_2^6a_3 + a_1^4a_3^4 + a_1^8. \end{aligned} \tag{15}$$

Taking $n = 2$, $m = 1$, we have in (15) an absolute invariant† of the binary cubic in the $GF[3^2]$.

Taking $n = 1$, $m = 2$, and reducing by $a^3 \equiv a$, we get

$$\begin{aligned} J = {} & a_0^2a_2^2 - a_0a_1a_2a_3 + a_0a_1^2a_2 + a_2^2 \\ & + a_1a_2^2a_3 + a_1^2a_3^2 + a_1^2 \pmod 3, \end{aligned} \tag{15′}$$

an absolute invariant of the binary cubic in the $GF[3]$.

* See (14), *Transactions*, l. c., p. 211.

† See (78), *Transactions*, l. c., p. 227; direct by (58), p. 222.

Now K and J, increased by unity, give (apart from a multiple of 3) the number of sets of values for which a cubic form (with the coefficients not all zero) vanishes in the $GF[3]$ and the $GF[3^2]$, respectively. We find that *

$$J = K + \Delta^2 - \Delta \quad (\Delta = \text{discriminant}),$$

$$K^2 + K = J^2 + J.$$

But K is not a rational function of J (in view of the first and second forms below), nor J a rational function of K (in view of the second and third forms):

Form.	K	J	Δ
$x^3 - xy^2 + y^3$	-1	-1	1
$x^3 + xy^2$	0	-1	-1
x^3	0	0	0
$x^2y + xy^2$	-1	-1	1
x^2y	1	1	0
Vanishing	0	0	0

Every cubic can be transformed modulo 3 into one of those given in the table (*Transactions*, l. c., page 232).

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NOTE ON JACOBI'S EQUATION IN THE CALCULUS OF VARIATIONS.

BY PROFESSOR MAX MASON.

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IN Weierstrass's theory of the calculus of variations † it is shown that the determinant

$$\omega = \frac{\partial y}{\partial t}\frac{\partial x}{\partial a} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial a}$$

formed from the equations $x = x(t, a)$, $y = y(t, a)$ of a family of extremals of the integral

* If we employ the invariant $P = \Delta + 1 - K$ (l. c., p. 211), we have

$$J = K^2 + K + P - 1.$$

† See for example Bolza, Lectures on the calculus of variations, Chicago, 1904.

$$J = \int F(x, y, x', y')dt$$

is a solution of Jacobi's equation

$$(\omega' F_1)' - \omega F_2 = 0.$$

This result, which is of fundamental importance in the theory, is obtained by differentiating the Euler equations of the extremals

$$F_x - \frac{d}{dt}F_{x'} = 0, \quad F_y - \frac{d}{dt}F_{y'} = 0$$

with respect to the parameter a, a method which involves considerable reckoning and the introduction of two sets of functions L, M, N; L_1, M_1, N_1, which serve to define F_2.

It is the object of this note to derive the result above stated directly from the single equation of the extremals

$$T \equiv F_{xy'} - F_{yx'} + F_1(x'y'' - y'x'') = 0, \tag{1}$$

which is equivalent to the pair of dependent Euler equations. The introduction of successive sets of auxiliary functions to define F_2 is in this way avoided, and an explicit form for F_2 is obtained.

Write for abbreviation $\partial x/\partial a = \xi$, $\partial y/\partial a = \eta$, and denote differentiation with respect to t by accents. Then

$$\omega = y'\xi - x'\eta, \quad \omega' = y''\xi - x''\eta + y'\xi' - x'\eta',$$
$$\omega'' = y'''\xi - x'''\eta + 2(y''\xi' - x''\eta') + y'\xi'' - x'\eta''.$$

If equation (1) be differentiated with respect to a, and the quantity $[y'''\xi - x'''\eta + 3(y''\xi' - x''\eta')]F_1$ be subtracted and added in the result, the following equation is obtained:

$$\begin{aligned} &- \omega'' F_1 + \xi'[3y''F_1 + (x'y'' - y'x'')F_{1x'} - x'y'F_{1x} - y'^2 F_{1y}] \\ &+ \eta'[-3x''F_1 + (x'y'' - y'x'')F_{1y'} + x'^2 F_{1x} + x'y'F_{1y}] \\ &+ \xi[y'''F_1 + (x'y'' - y'x'')F_{1x} + F_{xxy'} - F_{xyx'}] \\ &+ \eta[-x'''F_1 + (x'y'' - y'x'')F_{1y} + F_{xyy'} - F_{yyx'}] = 0. \end{aligned} \tag{2}$$

Since

$$F_1' = x''F_{1x'} + y''F_{1y'} + x'F_{1x} + y'F_{1y},$$

the coefficients of ξ' and η' are equal to

$$y''(3F_1 + x'F_{1x'} + y'F_{1y'}) - y'F_1',$$
$$-x''(3F_1 + x'F_{1x'} + y'F_{1y'}) + x'F_1',$$

respectively. Now it may be shown from the homogeneity property of F that

$$3F_1 + x'F_{1x'} + y'F_{1y'} = 0. \tag{3}$$

In fact, on differentiating the identity

$$x'F_{x'} + y'F_{y'} = F \tag{4}$$

twice with respect to x', the equation

$$F_{x'x'} + x'F_{x'x'x'} + y'F_{x'x'y'} = 0$$

is obtained. If the second derivatives be expressed in terms of F_1, this equation becomes

$$y'^2(3F_1 + x'F_{1x'} + y'F_{1y'}) = 0.$$

A similar equation, where the factor y'^2 is replaced by x'^2, is obtained by differentiating (4) with respect to y'. Since x' and y' are not simultaneously zero, equation (3) must hold. The coefficients of ξ' and η' in equation (2) are therefore $-y'F'$ and $x'F_1'$ respectively. After adding and subtracting the expression $(y''\xi - x''\eta)F_1'$, equation (2) takes the form

$$-(\omega'F_1)' + P\xi + Q\eta = 0, \tag{5}$$

where

$$\begin{aligned} P &= y'''F_1 + (x'y'' - y'x'')F_{1x} + F_{y'xx} - F_{x'xy} + y''F_1', \\ Q &= -x'''F_1 + (x'y'' - y'x'')F_{1y} + F_{y'xy} - F_{x'yy} - x''F_1'. \end{aligned} \tag{6}$$

Now

$$x'P + y'Q = \frac{d}{dt}T = 0,$$

so that there exists a function F_2 such that

$$P = y'F_2, \quad Q = -x'F_2. \tag{7}$$

Therefore, after changing the signs in equation (5) the desired equation

$$(\omega'F_1)' - \omega F_2 = 0$$

is obtained.

The function F_2 determined by equations (6) and (7) may be found explicitly from the equation

$$(x'^2 + y'^2)F_2 = y'P - x'Q.$$

On expanding the second member and collecting terms, this equation becomes

$$(x'^2 + y'^2)F_2 = (x'x''' + y'y''')F_1 + (x'x'' + y'y'')F_1'$$
$$-F_{xx'}' - F_{yy'}' + x'(F_{x'xx} + F_{x'yy}) + y'(F_{y'xx} + F_{y'yy}).$$

Now on differentiating the identity

$$x'F_{x'} + y'F_{y'} = F$$

twice with respect to x or y, the equations

$$x'F_{x'xx} + y'F_{y'xx} = F_{xx}, \quad x'F_{x'yy} + y'F_{y'yy} = F_{yy}$$

are obtained, so that F_2 is given by the equation

$$(x'^2 + y'^2)F_2 = (x'x''' + y'y''')F_1$$
$$+ (x'x'' + y'y'')F_1' + F_{xx} - F_{xx'}' + F_{yy} - F_{yy'}'.$$

In case the parameter t is the length of arc, so that $x'^2 + y'^2 \equiv 1$, the function F_2 has the simpler form

$$F_2 = F_{xx} - F_{xx'}' + F_{yy} - F_{yy'}' - (x''^2 + y''^2)F_1.$$

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ON THE DISTANCE FROM A POINT TO A SURFACE.

BY PROFESSOR E. R. HEDRICK.

(Read before the American Mathematical Society, September 5, 1907.)

THE discussion of the extrema of the distance from a point to a surface has been made the basis for the treatment of principal radii of curvature and for the classification of points on a surface by several writers.* In this connection it is interest-

* See, e. g., Goursat, Cours d'analyse, or English translation, no. 60; the statements there made are correct, the example here considered falling under the case $s^2 - rt = 0$. See also BULLETIN, vol. 13, no. 9, pp. 447, 448; the statements of this article differ in their spirit from those of the present article, and comparisons must be made with this understanding.

ing to notice that a variation of an example due to Peano * results in an example of a surface such that the distance from a certain point to the surface is at a minimum at a certain point of the surface for every normal plane section of the surface through that point, while the same distance is *not* at a minimum in general on the surface at the same point.

The surface

$$2x^4 - 3x^2y - x^2 - z^2 + 1 = 0 \tag{1}$$

is *regular* † at the point (0, 0, 1), the normal at that point is the axis of z, and the square of the distance from the origin to the surface is

$$D^2 = x^2 + y^2 + z^2 = 1 + (y - x^2)(y - 2x^2), \tag{2}$$

so that the positive determination of D in a sufficiently small circle about the origin in the (x, y) plane is real and does not vanish. An extremum of D^2 therefore corresponds precisely to the same extremum of D. But the value of D^2 given by (2) is nothing but the Peano function

$$F(x, y) = (y - x^2)(y - 2x^2) \tag{3}$$

increased by unity. It follows that D is at a minimum at $(x = 0, y = 0)$ along any normal section of the surface, *i. e.*, for $y = kx$ and for $x = 0$; but D is not at a minimum at (0, 0). For these are the known properties of $F(x, y)$. *That the distance from a point on the normal to each normal section is at a minimum (or at a maximum) for every normal section through the foot of the normal does not show that the same distance is at a minimum (maximum) for the surface itself.*

The surface

$$\begin{aligned} x^2 + z^2 - 1 + 3ye^{-1/x^2} - 2e^{-2/x^2} &= 0, \quad \text{for } x \neq 0; \\ x^2 + z^2 - 1 &= 0, \quad \text{for } x = 0; \end{aligned} \tag{4}$$

is continuous and regular at the point (0, 0, 1), the normal is the axis of z, and the square of the distance from the origin to the surface is

* A. Gennochi and G. Peano, Calcolo differenziale, etc., 1884, p. xxix; German translation, Bohlmann and Schepp, 1899, p. 332.

† That is the partial derivatives of the left-hand side of (1) exist and do not all vanish at that point. It is easy to construct and to visualize the surface (1).

$$(5)\qquad \begin{aligned} D^2 &= 1 + (y - e^{-1/x^2})(y - 2e^{-1/x^2}), \quad \text{when } x \neq 0; \\ D^2 &= 1 + y^2, \quad \text{when } x = 0; \end{aligned}$$

which is precisely the function

$$(6)\qquad \begin{aligned} \phi(x, y) &= (y - e^{-1/x^2})(y - 2e^{-2/x^2}), \quad \text{when } x \neq 0; \\ \phi(x, y) &= y^2, \quad \text{when } x = 0; \end{aligned}$$

increased by unity; the function $\phi(x, y)$ given by (6) has been studied *; it is at a minimum for any analytic curve in the (x, y) plane through the point (0, 0); but it is not at a minimum in the region about (0, 0). Thus even the knowledge that the distance from a point on the normal to a surface is at a minimum at the foot of the normal for every curve cut out of the surface by an analytic cylinder through the normal, does not prove that the same distance is at a minimum for the surface itself.

COLUMBIA, MO.,
January 11, 1908.

A GEOMETRIC REPRESENTATION OF THE GALOIS FIELD.

BY DR. L. I. NEIKIRK.

(Read before the Chicago Section of the American Mathematical Society, March 30, 1907.)

THE roots of irreducible congruences were first introduced into mathematics by Galois.† Various writers since then have contributed to the theory of irreducible congruences and classes of residues.‡ The greatest progress of recent years was made by Moore.§ He proved that any finite abstract field in which division is unique is an abstract form of the Galois field; that its order is the power of a prime p^n, and that for any order it is unique, being independent of the particular irreducible congruence of degree n used in defining it.

* *Annals of Mathematics*, vol. 8, no. 4 (July, 1907), pp. 172–174. The exact significance of "analytic," as here used, is there specified, and a more general statement of the property quoted is given.

† "Sur la théorie des nombres," *Bulletin des Sciences Math.* de M. Ferussac (1830); also Œuvres mathématiques d'Evariste Galois, Gauthier-Villars, Paris, 1897.

‡ See the preface of Linear Groups by L. E. Dickson.

§ BULLETIN, December, 1893; Chicago Congress Mathematical Papers, pp. 208–242.

More recently Maclagan-Wedderburn* and Dickson proved simultaneously that any finite field is necessarily commutative.

The object of this paper is to put the Galois field — and, by the above, any finite field — in the concrete form of a geometric representation.†

Consider the equation $f(x) = 0$ of degree n and irreducible in the domain of rational numbers. It has a root j which we may represent in the ordinary way on the complex plane. If we plot all the points

$$c_1 j^{n-1} + c_2 j^{n-2} + \cdots + c_{n-1} j + c_n, \tag{1}$$

the c's being integers, we get a complete geometric representation of the integers of the algebra defined by $f(x) = 0$. This set of points is closed with regard to the ordinary geometric forms of the rational operations addition, subtraction, multiplication, but not division.

The following theorem is due to Jacobi:‡

If the vectors $\omega_1, \omega_2, \cdots, \omega_r$ $(r > 2)$ are not connected by some relation of the form

$$m_1\omega_1 + m_2\omega_2 + \cdots + m_r\omega_r = 0,$$

then integral values of the m's can be chosen making

$$|m_1\omega_1 + m_2\omega_2 + \cdots + m_r\omega_r| < \epsilon,$$

where ϵ may be any arbitrarily small positive quantity.

By this theorem all the points of the set (1) for $n > 2$ are limiting points, and the set forms an enumerable assemblage, dense around its own points.

Consider the Galois field defined by the irreducible congruence $f(x) \equiv 0 \pmod{p}$.§ We apply to this the representation given by (1) by distinguishing two kinds of points, viz., reduced points for which the c's are positive and less than p, and the remaining points called congruent points.‖ In particular, the points

$$(\bar{c}_1 + k_1 p) j^{n-1} + (\bar{c}_2 + k_2 p) j^{n-2} + \cdots + (\bar{c}_{n-1} + k_{n-1} p) j + (\bar{c}_n + k_n p)$$

* *Transactions Amer. Math. Society*, vol. 6 (1905), pp. 349–352.

† The geometric representation of residues to a prime modulus $(n = 1)$ is given by Dickson, Linear Groups, p. 3, and will not be considered here.

‡ Ges. Werke, vol. 2, pp. 27–32; also see Clebsch and Gordan's Theorie der Abel'schen Functionen, § 38.

§ This constitutes a farther restriction on $f(x)$.

‖ The exceptional case $x^2 + x + 1 \equiv 0 \pmod{2}$ noted by Galois (Works, footnote p. 17) is only an apparent and not a real exception here.

are said to be congruent to each other and to the reduced point

$$\bar{c}_1 j^{n-1} + \bar{c}_2 j^{n-2} + \cdots + \bar{c}_{n-1} j + \bar{c}_n.$$

When two reduced points are combined by the geometric forms of addition, subtraction, and multiplication the result is in general a point of set (1), not a reduced point but congruent to some definite third reduced point. This gives a geometric interpretation of the above three rational operations in the Galois field.

Let ω be a primitive root among the reduced points. Consider the points

$$\omega, \omega^2, \cdots, \omega^{p^n-1} (\equiv 1); \tag{2}$$

no two of these are congruent points. The quotient of two of these, $\omega^s \div \omega^r = \omega^{s-r}$ $(s > r)$, gives a definite point in set (2). If $r > s$, it will be necessary to multiply ω^s by ω^{p^n-1} before performing the operation of division.

The geometric operation of division in the Galois field is performed in the following manner. Points in set (2) congruent to the two reduced points given are combined by the ordinary geometric form of division; the resulting point of set (2) is congruent to a definite third reduced point. This completes the four rational operations in the Galois field.

Only a limited number of points of set (1) are used in all possible operations between the reduced points. This geometric representation of the Galois field is not unique, as an indefinitely large number of choices of j may be made. In particular, we may always chose j real if we wish, as is always possible, to make the constant term of $f(x)$ negative.

If $j = \rho(\cos\theta + i \sin\theta) = \rho\epsilon^{i\theta}$, then the elements of the Galois field are given in the form

$$\bar{c}_1 \rho^{n-1} \epsilon^{i(n-1)\theta} + \bar{c}_2 \rho^{n-2} \epsilon^{i(n-2)\theta} + \cdots + \bar{c}_{n-1} \rho \epsilon^{i\theta} + \bar{c}_n.$$

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CONCERNING THE DEGREE OF AN IRREDUCIBLE LINEAR HOMOGENEOUS GROUP.

[BY PROFESSOR W. B. FITE.

(Read before the American Mathematical Society, September 6, 1907.)

IN the *Transactions*, volume 8 (1907), pages 107–112, the writer has considered the connection between the degree of an irreducible linear homogeneous group and its abstract group properties. The discussion was limited for the most part to groups whose orders are powers of a prime. In the present paper some of the results of the former paper are extended to other groups.

THEOREM I. *A linear homogeneous group G all of whose invariant operations are similarity substitutions and such that every invariant subgroup contains invariant operations besides identity either is irreducible or is simply isomorphic with each of its irreducible components.*

If G is reducible, suppose that it has been put into its completely reduced form. The invariant operations will not be affected by this change. If any irreducible component were not simply isomorphic with G, the subgroup of G that would correspond to identity in this component would contain invariant operations besides identity. But this is impossible since every invariant operation of G is a similarity substitution. Hence every irreducible component is simply isomorphic with G.

Let G be any group of finite order g that has a cyclic central generated by the operation h of order a (>1) and when G is written as a regular permutation group, of form

$$(x_{1,1}x_{1,2}\cdots x_{1,a})(x_{2,1}x_{2,2}\cdots x_{2,a})\cdots(x_{g/a,1}x_{g/a,2}\cdots x_{g/a,a}).$$

The linear substitution S

$$y_{i,j}=\sum_{k=1}^{a}\omega^{(k-1)(j-1)}x_{i,k}\quad(i=1,2,\cdots,g/a;\ j=1,2,\cdots,a),$$

where ω is a primitive ath root of unity, transforms G into a semi-canonical form and transforms h into its normal form.*

* *Transactions Amer. Math. Society*, loc. cit., p. 108.

In this transformed form of G the variables

$$x_{1,j}, x_{2,j}, \cdots, x_{g/a,j} \qquad (j = 1, 2, \cdots, a)$$

are transformed into linear combinations of themselves. Let G_j be the group formed by the substitutions of G as far as they affect these g/a variables. If $j-1$ is relatively prime to a, G_j is simply isomorphic with G.

If we suppose that every invariant subgroup of G contains at least one invariant operation besides identity, it follows that G_j is simply isomorphic with each of its irreducible components. We have therefore

THEOREM II. *A necessary and sufficient condition that a group all of whose invariant subgroups contain invariant operations besides identity be simply isomorphic with an irreducible group is that its central be cyclic.*

If $j'-1$ $(j' \neq j)$ is also relatively prime to a, then since the coefficients of G_j are either zero or powers of ω^{j-1} and $G_{j'}$ differs from G_j only in having $\omega^{j'-1}$ for ω^{j-1}, it follows that for every irreducible component of G_j there is a non-equivalent irreducible component of $G_{j'}$, and conversely. Moreover if $j-1$ is not relatively prime to a, G_j is not simply isomorphic with G. Hence the number of irreducible representations of G of any degree that are simply isomorphic with it is a multiple of $\phi(a)$.

If G is simply isomorphic with an irreducible group of the maximum degree * $n = \sqrt{g/a}$, then every invariant subgroup of G contains invariant operations besides identity. This follows from the fact that when the regular form of G is completely reduced every irreducible component occurs a number of times equal to its degree.†. Moreover the number of distinct representations of G that are simply isomorphic with it is $\phi(a)$.

Consider the formula ‡

$$\sum_R \chi(S^{-1}R^{-1}SR) = \frac{g}{f}\chi(S)\chi(S^{-1}).$$

Now suppose that G is a group in which every non-invariant commutator gives an invariant commutator besides identity,§

* *Transactions*, vol. 7 (1906), p. 67.

† Burnside, *Acta Mathematica*, vol. 28 (1904), p. 383.

‡ Frobenius, *Berliner Sitzungberichte*, 1896, II., p. 1362. We here indicate the order of the group by g instead of h.

§ This category of groups is contained in the one defined by the property that every invariant subgroup contains invariant operations besides identity.

and let S be a non-invariant operation of G that gives no invariant commutator besides identity. It follows from the formula just given that $\chi(S) \neq 0$. That is, in an irreducible group of the kind just described a necessary and sufficient condition that the trace (Spur) of any operation be different from zero is that the operation give no invariant commutator besides identity.*

Since the trace of every non-invariant commutator is zero and S gives no invariant commutators besides identity, we have $sf^2/g = \chi(S)\chi(S^{-1})$, where s indicates the number of operations with which S is commutative. Now

$$\sum_S \chi(S)\chi(S^{-1}) = g.\dagger$$

This gives $f^2 = g^2/(hg + \Sigma s_i)$, where h indicates the number of invariant operations of G, and s_i indicates the number of operations commutative with S_i (S_1, S_2, $\cdots$ being the non-invariant operations that give no invariant commutators besides identity). But each of the g/s_i operations of the conjugate set to which S_i belongs gives the same s_i. Therefore $f^2 = g/(h + j)$, where j is the number of sets of conjugates among the non-invariant operations of G that give no invariant commutator besides identity. This formula gives, for example, the degree of any irreducible group of the third class.

If G is of order p^m (p a prime), then $\chi(S)\chi(S^{-1}) = p^{r+2n-m}$, where S is a non-invariant operation that gives no invariant commutators besides identity, p^r is the number of operations commutative with S, and p^n is the degree of G. Hence, *in an irreducible group of order p^m such that every non-invariant commutator gives an invariant commutator besides identity, the square of the absolute value of the trace of any operation is a power of p (positive or negative).*‡

If G is any irreducible group of order p^m we have §

$$\sum_i x_i p^{2n_i} = p^{m-1}(p - 1).$$

But $\sum_i x_i$ is the excess of the number of conjugate sets of G over the number of conjugate sets of G/H. Hence

* Cf. *Transactions*, vol. 8 (1907), p. 110.
† Frobenius, loc. cit., pp. 1362, 1372.
‡ Cf. *Transactions*, loc. cit., p. 110.
§ Loc. cit., p. 109.

$$\sum_i x_i = \frac{(p^a + j)(p-1)}{p}.$$

If now G can be simply isomorphic with irreducible groups of only one degree p^a, then

$$\frac{(p^a + j)(p-1)}{p} p^{2a} = p^{m-1}(p-1), \qquad p^{2a} = \frac{p^m}{p^a + j}.$$

Suppose now that G is simply isomorphic with irreducible groups of just two different degrees, p^{a_1} and p^{a_2} $(n_1 < n_2)$. Then

$$x_1 p^{2a_1} + x_2 p^{2a_2} = p^{m-1}(p-1),$$

$$x_1 + x_2 = \frac{(p^a + j)(p-1)}{p}.$$

Since x_1 and x_2 are integers, $p^a + j$ must be a power of p, say p^r. Obviously p^{2a_1} and p^{2a_2} cannot both be less than, or both greater than p^{m-r}. Therefore let

$$p^{2a_1} = p^{m-r-t_1} \quad \text{and} \quad p^{2a_2} = p^{m-r+t_2} \qquad (t_1, t_2 > 0).$$

Then

$$x_2 = \frac{p^{r-1}(p-1)(p^{t_1} - 1)}{p^{t_1+t_2} - 1}.$$

But this cannot be an integer. Hence, *a group of order a power of a prime cannot be simply isomorphic with irreducible groups of just two different degrees.*

It follows from this that if a group of order p^m is simply isomorphic with irreducible groups of different degrees, m must be greater than 9.*

CORNELL UNIVERSITY,
January, 1908.

*See *Transactions*, loc. cit., p. 110.

ON THE LORENTZIAN TRANSFORMATION AND THE RADIATION FROM A MOVING ELECTRON.

BY DR. F. R. SHARPE.

(Read before the American Mathematical Society, February 29, 1908.)

1. *Introduction.* The electromagnetic field produced by a moving charge has been discussed by many writers.* Langevin † in particular has given an elegant investigation of the field due to a moving electron. Poincaré ‡ has partially solved Langevin's problem by the Lorentzian § transformation. In the present paper the Lorentzian transformation is applied to the expressions for the scalar and vector potentials in the form of definite integrals instead of to the differential equations for the electromagnetic field. The field due to an electron whose velocity is zero at the instant considered is very simply determined. By means of the Lorentzian transformation, the field for the more general case when the velocity is not zero is deduced. The results are Langevin's expression for the electric force and a new expression for the magnetic force.

2. *The electromagnetic equations and their solution.* Let E denote the electric force, H the magnetic force, ρ the volume density, and v the velocity of the electrons, and let the velocity of light be the unit of velocity. Lorentz's || fundamental equations are

$$(1) \qquad \operatorname{div} E = \rho, \qquad (2) \qquad \operatorname{div} \rho v = -\frac{\partial \rho}{\partial t},$$

$$(3) \qquad \operatorname{curl} H = \frac{\partial E}{\partial t} + \rho v, \qquad (4) \qquad \operatorname{curl} E = -\frac{\partial H}{\partial t}.$$

The solution of these equations is known to be reducible to the determination of a scalar potential ϕ and a vector potential α which satisfy the equations

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*See Lorentz, Encyklopädie, d. math. Wissensch., vol. V. 2, p. 174.
†*Journal de Physique*, 1905.
‡ *Circolo mat. di Palermo*, 1906.
§ Lorentz, *Amsterdam Proc.*, 1903. Einstein, *Annalen der Physik*, 1905.
|| Loc. cit.

$$(5)\quad \nabla^2\phi - \frac{\partial^2\phi}{\partial t^2} = -\rho, \qquad (6)\quad \nabla^2\alpha - \frac{\partial^2\alpha}{\partial t^2} = -\rho v,$$

where

$$(7)\quad E = -\frac{\partial\alpha}{\partial t} - \text{grad}\ \phi \qquad \text{and} \qquad (8)\quad H = \text{curl}\ \alpha.$$

The solutions of (5) and (6) are known to be *

$$(9)\qquad \phi = \frac{1}{4\pi}\int\int\int \frac{\rho_1 dx_1 dy_1 dz_1}{r},$$

$$(10)\qquad \alpha = \frac{1}{4\pi}\int\int\int \frac{\rho_1 v_1 dx_1 dy_1 dz_1}{r},$$

where ρ_1 and v_1 are the values of ρ and v at (x_1, y_1, z_1) at time $t - r$, r being the time from (x, y, z) to (x_1, y_1, z_1). These expressions for the potentials may be expressed in terms of the actual charges of the electrons. If we imagine a sphere whose center is (x, y, z) and whose radius r diminishes with the velocity of light and is zero at time t, an electron will affect (x, y, z) only while the surface of the sphere is passing through the electron. Hence if the radial velocity of an electron is v_r, the effect of the motion of the electron is to increase the apparent charge in the ratio 1 to $1 - v_r$. Therefore if de denotes the actual charge of an electron at (x_1, y_1, z_1),

$$(11)\quad \phi = \int \frac{de}{r(1 - v_r)}, \qquad (12)\quad \alpha = \int \frac{v de}{r(1 - v_r)}.$$

3. *The Lorentzian Transformation.* Consider the transformation

$$(13)\quad x' = \beta(x - Vt), \quad t' = \beta(t - Vx), \qquad (\beta = 1/\sqrt{1 - V^2})$$

and its inverse

$$(14)\qquad x = \beta(x' + Vt'), \quad t = \beta(t' + Vx').$$

If we wish to transform the electrical system into a new electrical system with the same charges, we must have

$$(15)\qquad \int\int\int \rho' dx' dy dz = \int\int\int \rho dx dy dz.$$

* Rayleigh, Theory of sound, vol. 2, p. 104.

On the left hand side of this equation t' is constant, while on the other side t is constant. On account of the velocity v_x of the electrons $(x+dx+v_x dt, t+dt)$ is tranformed into $(x'+dx', t')$. Hence from (13)

$$dx' = \beta(dx + v_x dt - Vdt), \quad 0 = \beta\{dt - V(dx + v_x dt)\}.$$

Solving, we find

$$dx' = \frac{dx}{\beta(1 - Vv_x)}$$

and therefore to satisfy (15) we must have

$$\rho' = \beta\rho(1 - Vv_x). \tag{16}$$

It is important to notice that (13) and (16) completely determine the new electrical system.

We proceed to find the values of v', ϕ', α', E', H'. Differentiating (13) we have

$$v'_x = \frac{dx'}{dt'} = \frac{v_x - V}{1 - Vv_x}, \qquad v'_y = \frac{dy}{dt'} = \frac{v_y}{\beta(1 - Vv_x)},$$

$$v'_z = \frac{dz}{dt'} = \frac{v_z}{\beta(1 - Vv_x)}. \tag{17}$$

It is easily verified that the values of ρ' and v' just obtained satisfy the continuity equation

$$\text{div } \rho' v' = -\frac{\partial \rho'}{\partial t'}. \tag{2'}$$

It should also be noticed that in (11) and (12)

$$t - t_1 = r \tag{18}$$

or

$$(t - t_1)^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2.$$

This equation is transformed by (14) into

$$(t' - t'_1)^2 = (x' - x'_1)^2 + (y - y_1)^2 + (z - z_1)^2$$

or

$$t' - t'_1 = r'. \tag{18'}$$

Hence the new electron is in position at time t'_1 to act on the

new point (x', y', z') at time t'. Further it is readily shown from (17) and (18) that

$$r'(1-v_r') = \frac{r(1-v_r)}{\beta(1-Vv_x)}. \tag{19}$$

Hence

$$\phi' = \int \frac{de}{r'(1-v_r')} = \int \frac{\beta(1-Vv_x)de}{r(1-v_r)}$$

or

$$\phi' = \beta(\phi - V\alpha_x), \tag{20}$$

and similarly

$$\alpha_x' = \beta(\alpha_x - V\phi), \quad \alpha_y' = \alpha_y, \quad \alpha_z' = \alpha_z.$$

Again from (14)

$$\frac{\partial}{\partial x'} = \beta\left(\frac{\partial}{\partial x} + V\frac{\partial}{\partial t}\right), \quad \frac{\partial}{\partial t'} = \beta\left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial x}\right).$$

Therefore

$$E_x' = -\frac{\partial \alpha_x'}{\partial t} - \frac{\partial \phi'}{\partial x'} = -\frac{\partial \alpha_x}{\partial t} - \frac{\partial \phi}{\partial x}$$

or

$$E_x' = E_x, \tag{21}$$

and similarly

$$E_{y'} = \beta(E_y - VH_z), \quad E_{z'} = \beta(E_z + VH_y), \quad H_{x'} = H_x,$$
$$H_{y'} = \beta(H_y + VE_z), \quad H_{z'} = \beta(H_z - VE_y).$$

The new electrical system is therefore completely expressed in terms of the old system.

4. *Radiation from an electron.* If the charge of the electron is e the potentials are, from (11) and (12),

$$\phi = \frac{e}{4\pi r(1-v_r)}, \quad \alpha = \frac{ev}{4\pi r(1-v_r)}. \tag{22}$$

Langevin has calculated the derivatives of ϕ and α, and so determined the electric and magnetic forces. The problem is simplified by taking the x axis in the direction of the velocity and applying the Lorentzian transformation so as to reduce the velocity to zero; solving the new system and reversing the

transformation so as to obtain the solution of the original system. In this case

$$V = v_x, \quad v_y = v_z = v_x' = v_y' = v_z' = 0, \tag{23}$$

and from (22)

$$\phi' = \frac{e}{4\pi r'}, \quad \alpha' = 0. \tag{24}$$

The derivatives of ϕ' and α' depend on the acceleration j. On differentiating (17) and using (23), we obtain

$$j_x' = \frac{dv_x'}{dt'} = \beta^3 j_x, \quad j_y' = \beta^2 j_y, \quad j_z' = \beta^2 j_z. \tag{25}$$

From (22)

$$\phi' + d\phi' = \frac{e}{4\pi(r' + dr')(1 - j_r' dt_1')}, \quad d\alpha' = \frac{ej' dt_1'}{4\pi r'}.$$

But if t' is constant, (18) gives $dt_1' = -dr'$.
Hence

$$\text{grad } \phi' = -\frac{e}{4\pi r'^3}\bar{r}' - \frac{ej_r'}{4\pi r'^2}\bar{r}'.$$

Also if r' is constant $dt_1' = dt'$. Hence

$$\frac{\partial \alpha'}{\partial t'} = \frac{ej'}{4\pi r'}.$$

Hence

$$E' = \frac{e}{4\pi r'^3}\bar{r}' + \frac{ej_r'}{4\pi r'^2}\bar{r}' - \frac{\bar{e}j'}{4\pi r'}.$$

Again

$$H' = \text{curl } \alpha' = \frac{\partial \alpha_t'}{\partial r'} = \frac{e}{4\pi r'^2}[j'r'],$$

where $d\alpha_t'$ is the transverse component of $d\alpha'$. The first term of E' is the force E_1' due to the electron at rest. The other two terms are due to the acceleration and have a resultant $ej_t'/4\pi r' = E_2'$. This is obviously equal and perpendicular to H'. Hence E_2' and H' are equal and r', E_2', H' are mutually perpendicular, E_2' being in the plane of r' and j'. If we now apply the inverse Lorentzian transformation the first term of E' gives the well-known field for a uniformly moving charge,

the velocity wave of Langevin. The terms depending on the acceleration (when expressed in terms of the original coordinates the origin being taken at the electron so that $x_1 = y_1 = z_1 = 0$) are

$$E'_x = \frac{e}{4\pi(r - Vx)^3}\{(yj_y + zj_z)(x - Vr) - (y^2 + z^2)j_x\},$$

$$E'_y = \frac{e}{4\pi\beta(r - Vx)^3}[\{zj_z + \beta^2(x - Vr)j_x\}y - \{z^2 + \beta^2(x - Vr)^2\}j_y],$$

$$E'_z = \frac{e}{4\pi\beta(r - Vx)^3}[\{\beta^2(x - Vr)j_x + yj_y\}z - \{\beta^2(x - Vr)^2 + y^2\}j_z],$$

$$H'_x = \frac{e}{4\pi(r - Vx)^2}(zj_y - yj_z), \qquad (26)$$

$$H'_y = \frac{e\beta}{4\pi(r - Vx)^2}\{(x - Vr)j_z - zj_x\},$$

$$H'_z = \frac{e\beta}{4\pi(r - Vx)^2}\{yj_x - (x - Vr)j_y\}.$$

Making the transformation, we easily find

$$E_x = \frac{e}{4\pi(r - Vx)^3}\{(xj_x + yj_y + zj_z)(x - Vr) - r(r - Vx)j_x\},$$

$$E_y = \frac{e}{4\pi(r - Vx)^3}[(xj_x + yj_y + zj_z)y - r(r - Vx)j_y],$$

$$E_z = \frac{e}{4\pi(r - Vx)^3}\{(xj_x + yj_y + zj_z)z - r(r - Vx)j_z\}, \qquad (27)$$

$$H_x = \frac{e}{4\pi(r - Vx)^3}(zj_y - yj_z)(r - Vx),$$

$$H_y = \frac{e}{4\pi(r - Vx)^3}\{(xj_z - zj_x)(r - Vx) - (xj_x + yj_y + zj_z)Vz\}.$$

$$H_z = \frac{e}{4\pi(r - Vx)^3}\{(yj_x - xj_y)(r - Vx) + (xj_x + yj_y + zj_z)Vy\}.$$

Expressed in vector notation, we have therefore

$$E = \frac{e}{4\pi\{r - (\overline{Vr})\}^3}(rj)\overline{R} - \frac{er}{4\pi\{r - (\overline{Vr})\}^2}\dot{\overline{j}},$$

which is Langevin's result, R being the distance from (x, y, z) to the point which the electron would reach at time t if its acceleration were zero; that is, a force directed towards this point and a force in the direction of the acceleration. Moreover

$$H = \frac{e}{4\pi\{r - (Vr)\}^3}[rj] - \frac{e}{4\pi\{r - (Vr)\}^3}(rj)[Vr],$$

that is, a force perpendicular to $\bar{r}$ and $\bar{j}$ and a force perpendicular to $\bar{r}$ and $\bar{V}$. It is easily verified from (27) that E is equal to H and that r, E, H are mutually perpendicular.

CORNELL UNIVERSITY,
January, 1908.

SHORTER NOTICES.

On the Resolution of Higher Singularities of Algebraic Curves into Ordinary Nodes. By B. M. WALKER. Doctor Dissertation, University of Chicago, 1906. 8vo. 52 pp. + Vita.

As stated in the introduction, this dissertation completes in detail a procedure proposed by Clebsch for resolving the higher singularities of algebraic plane curves. The method is this: To relate the points of the plane, one to one, to points of a general cubic surface in such a way that one point of multiplicity higher than 2, with tangents all distinct, is distributed into ordinary points of a curve on the cubic. Then by projecting back upon the plane from a center on the cubic surface, no new singularities are introduced except ordinary double points.

The first half of the work (26 pages) is devoted to the relation of plane and cubic surface by means of a three-parameter linear system of plane cubics with six common base points. The restriction that these six points shall not lie on a conic suffices to insure that there shall be in the plane no fundamental curves, *i. e.* no curves all of whose points correspond to a single point of the cubic. On the cubic surface, however, the six base points of the plane are represented by six straight lines. The author shows in detail what plane curves give rise to the rest of the twenty-seven lines on the cubic. Six of these are conics which contain five of the base points apiece; the

others are the fifteen join lines of the six base points. This study is essential in preparation for the topics which follow.

In the latter half appears the main object of this research. In the singular point to be resolved is placed one of the six base points of the plane cubics. Thus on the surface this r-fold point appears as r ordinary points on the corresponding fundamental line. Conscientiously it is proved that other multiple points do not change their essential character by the transformation. For this purpose Jordan's definition of cycle of a plane curve is happily generalized. Then are specified the five loci on the cubic surface which must be avoided in choosing a center for the projection back on the plane, and it is proved that these restrictions are sufficient for the exclusion of new higher singularities.

It should be noted that Dr. Walker takes his start from Noether's reduction of the curve to one at whose singular points the tangents are all distinct. This work is a thoroughly readable presentation of what is probably the most graphic reduction of the higher singularities.

H. S. WHITE.

Die Kegelschnitte des Gregorius a St. Vincentio in vergleichender Bearbeitung. Von KARL BOPP. Mit 329 Textfiguren. *Abhandlungen zur Geschichte der mathematischen Wissenschaften*, XX Heft, 2 Stück, pp. 83–314. Leipzig, B. G. Teubner, 1907.

THE glare of notoriety of Gregorius a St. Vincentio as a circle-squarer has cast his solid achievements into obscurity. At a time when only four of the seven books of the conics of Apollonius of Perga were known in the occident, Gregorius prepared a masterly work on conic sections, the Opus geometricum, which was published at Antwerp in 1647. He did not possess the genius of his great contemporaries, Desargues and Pascal. Yet in the endeavor to gain a detailed picture of the progress of geometry of the seventeenth century, one cannot overlook Gregorius a St. Vincentio. Through the researches of Bosmans there has been an enrichment of biographical detail of this man. To make a knowledge of his very diffuse treatise more readily accessible and to point out his relation to ancient writers, as well as his influence on the progress of geometry, Mr. Bopp has prepared the booklet under review. The chief novelty in Gregorius's geometry is the method of transformation

of one conic into another, called *per subtensas* (by chords), which, in the opinion of Bopp, "even now would be serviceable for didactic purposes" (page 307), which "holds *in nuce* the entire analytic geometry of conics" (page 295) and "is fit to place him, Gregorius, among the founders of analytic geometry" (page 309). While, in our opinion, Bopp claims too much for this method, it is doubtless of historic interest. To gain an idea of it, take the following example: In subjecting the parabola to the transformation "by chords," Gregorius in one place draws a chord from the vertex to any point of the parabola and takes the length of this chord as the ordinate of a point having the same abscissa as the chosen point on the parabola. This new point lies on the equilateral hyperbola whose transverse axis lies in the geometric axis and is equal to the latus rectum of the parabola. By this process the hyperbola can be constructed by points. It is shown how to derive, *per subtensas*, the hyperbola from the ellipse and also how the hyperbola may be transformed into itself. Similarly for the other conics.

FLORIAN CAJORI.

Annuaire du Bureau des Longitudes pour l'An 1908. Paris, Gauthier-Villars.

THE *Annuaire* has been published continuously since 1796, and the present volume is the 112th of the series. This publication has been so often reviewed in the "Shorter Notices" that a notice of the special features for the current year is sufficient. In accordance with the plan adopted in 1904, it contains the tables and explanations of physical and chemical constants, those of geographical and statistical constants being inserted only in the odd-numbered years.

The special articles at the end of the volume are five in number. The first and longest is a popular account, by M. G. Bigourdan, of the methods used for obtaining the parallaxes and distances of the heavenly bodies. The author adopts a chronological treatment and gives a fairly complete historical summary without at any time wearying the reader. M. Deslandres gives an account of the meetings at St. Louis, Oxford, and Meudon of the international union for solar research, from which it is easy to see how the systematic study of a branch of physics may elevate the subject into a science. The observatory of Montsouris in France exists for educational purposes only: M. Guyon explains its methods and equipment. Many

geographers and travellers have learnt there the practical astronomy necessary to those who explore far from the beaten tracks. One of its features is the absence of any regular classes; those who go there to learn can have a lesson at any time and the lesson is made to suit the special needs of the pupil. The last two articles are obituary notices of M. Loewy by H. Poincaré; and of C. Trépied by M. Loewy, written shortly before the sudden death of the latter.

ERNEST W. BROWN.

Die Zustandsgleichung der Gase und Flüssigkeiten und die Kontinuitätstheorie. Von J. P. KUENEN. Braunschweig, Vieweg und Sohn, 1907. x + 241 pp.

KUENEN'S volume on the equation of state of gases and fluids is one of the twenty monographs already printed by Vieweg and Son under the general title "Die Wissenschaft." It must be obvious to all that the subject of this particular volume is one which lends itself well to treatment in a separate monograph. The main outlines of the theory of corresponding states are given in many books, and no book which has to do with gases or fluids can get on without the equation of state; but the details of those theories and their agreement or disagreement with the results of the hundreds of experiments which have been performed since they were broached are not to be found collected in general texts and require for their satisfactory treatment a monograph like this.

The first five chapters, of about ten pages each, may be said to be of a heuristic and qualitative nature. The author traces the history of the rise of observations on the phenomena of condensation and on the existence of some principle of continuity between the different states of matter. He touches upon the kinetic theory sufficiently to show the justification of Boyle's law from that point of view and to indicate how van der Waals was led to his equation. After obtaining that equation, the author goes on to a careful explanation of phenomena of condensation and of the principle of continuity from the basis furnished by the equation. At about this point there begin to appear numerous evidences of the great care with which matters are to be set forth in their true light, and of the conscientious criticism with which the author is to expound the relation between experiment and theory. For instance, it is pointed out that although van der Waals's equation may be de-

duced from molecular considerations, yet in a broad way the theory of continuity does not depend on such considerations; it is noted that the vapor pressure is not strictly constant but has been found to depend on the relative amounts of liquid and vapor which are present; and, moreover, the effects of gravity and slight impurities are discussed and found to be surprisingly large for states of the mixture near the critical point.

The next four chapters fill about eighty pages and may be considered as the main part of the text. They deal with the correlation of the equation of state with experiment. Citations from experimental researches and deductions from thermodynamic theory as applied to van der Waals's equation are abundant and are freely intermingled until the reader cannot fail to see their interrelations, their points of concord, and their remaining outstanding differences. The critical constants are derived by two methods and put into relation with the theory of corresponding states. The wide divergence of the theoretical value of $RT/pv = 2.67$ at the critical point and actual values found by Young is amply illustrated by giving Young's results in detail. The influence of the temperature on the "constants" of van der Waals's equation is discussed, and some very neat methods (with figures) are given for readily comparing the theoretical and experimental isothermal curves. For handling the phenomena of saturation several distinct methods, including those of free energy and the thermodynamic potential, are offered and somewhat extensive tables afford a means of checking the results against actual observations. The chapter on thermal quantities (thermische Grössen) is equally detailed and instructive; there is really nothing more to be said about it.

The tenth chapter is a note of two pages on the dimensions of molecules. The following chapter deals with corresponding states. The distinction between substances which are normal and abnormal is brought out, and a number of somewhat unusual topics such as the determination of the degree of association, viscosity, capillarity, and molecular refraction are included in the discussion. The remaining three chapters are concerned with the improvements which have been suggested for van der Waals's equation, and the allied considerations. Among others, the equations of Lorentz, Reinganum, and Clausius are developed. The last of these chapters bears the title "Mathematical methods of deriving the equation of state"

and is naturally the most mathematical and difficult of the whole book. It should be stated, however, that the author nowhere shuns the use of that small amount of mathematics which is really essential to the proper development of his subject.

From this brief summary, it may be gathered that the range of topics which is treated is not large but that the treatment given is thorough, highly competent, open minded, and impartial. In short the book is just what it claims to be—a monograph in which the most important if not all of our knowledge, whether theoretical or experimental, on the equation of state is collected and carefully edited. For those who are interested in this subject the book appears to be well nigh indispensable, and for those who are not yet interested it would offer a pleasant day's reading in one of the most entertaining fields of modern physics. It would not do to close this review without mentioning the extensive bibliographical lists which follow many of the chapters. These will save the student from many unhappy hours spent in trying to find the important literature of the subject.

E. B. WILSON.

NOTES.

THE Fifteenth Summer Meeting of the AMERICAN MATHEMATICAL SOCIETY will be held at the University of Illinois, on Thursday and Friday, September 10–11, 1908.

THE following additional associate editors of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY have been appointed: Professor G. A. BLISS, of Princeton University; Professor F. R. MOULTON, of the University of Chicago; Professor E. J. WILCZYNSKI, of the University of Illinois.

AT the meeting of the London mathematical society held on February 13, the following papers were read: By H. A. DE S. PITTARD, "Proof that every algebraic equation has a root"; by W. H. YOUNG, "On the uniform approach of a continuous function to its limit"; by F. H. JACKSON, "Note on q-differences"; by A. E. WESTEN, "An extension of Eisenstein's law

of reciprocity (second paper)"; by E. CUNNINGHAM, "Conformal representation and the transformation of Laplace's equation."

A REGULAR meeting of the Syracuse section of the association of teachers of mathematics in the middle states and Maryland was held at the Syracuse high school, February 22. After a word of greeting by Principal HARPER, the following papers were read: By W. G. RAPPLEYE, "The teaching of elementary mathematics from a rational standpoint"; by G. R. STALEY, "The purpose of problems in plane geometry." The following officers were elected for the current year: President, W. G. RAPPLEYE; secretary and treasurer, D. PRATT. The time and place of the next meeting have not yet been definitely fixed. Sixty-nine names were added to the list of members.

THE subject of Professor POINCARÉ's conference at the Rome congress has been announced as "Les mathématiques de l'avenir." On account of his health, Professor KLEIN has decided not to attend the congress. His address "Ueber die mathematische Encyklopädie," will be delivered by Professor v. DYCK.

THE following Americans expect to attend the Rome Congress: E. W. Davis, T. S. Fiske, A. B. Frizell, J. G. Hardy, C. L. E. Moore, E. H. Moore, C. A. Noble, G. D. Olds, D. E. Smith, J. M. Van Vleck, and W. D. A. Westfall.

THE committee on methods of instruction which was proposed at the Dresden meeting of the Mathematiker-Vereinigung to investigate and rearrange the courses of study in the German secondary schools has now been organized and a definite plan of procedure adopted. The committee, which consists of one or two representatives of nearly every scientific society of Germany, will consider the whole field of instruction in mathematics and the natural sciences.

THE next meeting of the British association for the advancement of science will be held at Dublin, September 2–9, under the presidency of Mr. FRANCIS DARWIN. Dr. W. N. SHAW is chairman of section A, mathematics and physics.

THE academy of sciences of Vienna has advanced M. 1000 to assist in the publication of the writings of the late Professor L. BOLTZMANN.

In order to obtain as complete an edition of the mathematical works of Paolo Ruffini as possible, a general request is expressed that any letters, documents, and notices of every kind, that relate to his life, be forwarded to Professor B. G. Guccia in Palermo.

On the occasion of the dedication of the new buildings of the physical society of Frankfurt it was announced that the heirs of the late Eugen Tornow had presented to the academy (Akademie für Sozial- und Handelswissenschaften) the sum of 470,000 marks, to be used for advancing instruction in mathematics and the natural sciences.

The publishing house of B. G. Teubner in Leipzig announces that the second edition of the German translation of Pascal's Repertorium der höheren Mathematik is in the press. The first volume, which will soon appear, has been fundamentally revised with the assistance of the author and various other mathematicians under the general editorship of Dr. P. Epstein, of the University of Strassburg.

During the year 1907 the following candidates received the degree of doctor of science with mathematics as the major subject at the University of Paris: M. Faton, "Séries trigonométriques et série de Taylor"; M. Traynard, "Sur les fonctions théta de deux variables et les surfaces hyperelliptiques"; M. Montel, "Sur les suites infinies de fonctions"; M. Lambert, "Sur les coefficients du développement de la fonction perturbatrice."

The following advanced courses in mathematics are announced for the academic year 1908–1909:

Columbia University. — By Professor T. S. Fiske: Advanced calculus; introduction to the theory of functions of a real variable, three hours; Functions defined by linear differential equations, three hours. — By Professor F. N. Cole: Introduction to the theory of functions, three hours; Theory of plane curves, three hours. — By Professor James Maclay: Elliptic functions, three hours; Application of the calculus to the theory of surfaces and curves in space, three hours. — By Professor D. E. Smith: History of mathematics, two hours. — By Professor C. J. Keyser: The principles of mathematics, three hours; Modern theories in geometry, three

hours. — By Professor H. B. MITCHELL: Differential equations, two hours; Geometrical analysis, three hours. — By Professor Edward KASNER: Geometry of dynamical systems, two hours. — By Dr. G. H. LING: Theory of numbers, three hours, first half year; Advanced theory of numbers, three hours, second half year.

HARVARD UNIVERSITY. — By Professor W. E. BYERLY: Introduction to modern geometry and modern algebra, three hours; Trigonometric series (with Professor Peirce), three hours. — By Professor B. O. PEIRCE: Methods in mathematical physics, Elasticity, two hours. — By Professor W. F. OSGOOD: Differential and integral calculus (second course), three hours; Infinite series and products (first half year), three hours; Galois's theory of equations (second half year), three hours; Theory of functions (advanced course). — By Professor M. BÔCHER: Theory of functions (introductory course) three hours; The linear differential equations of physics, three hours. — By Professor C. L. BOUTON: Hydromechanics (second half year), three hours; Differential equations, Lie's theory of continuous groups, three hours. — By Professor J. K. WHITTEMORE: Elements of mechanics, three hours; Differential geometry of curves and surfaces (first half year), three hours. — By Professor E. V. HUNTINGTON: The fundamental concepts of mathematics, three hours. — By Dr. J. L. COOLIDGE: Line geometry (first half year) three hours. — By Dr. H. N. DAVIS: Dynamics of a rigid body, three hours.

Courses of reading and research are offered by Professors BYERLY, OSGOOD, BÔCHER, BOUTON, and WHITTEMORE; and a seminary in geometry will be conducted by Professor BOUTON, Professor WHITTEMORE, and Dr. COOLIDGE during the second half year.

PRINCETON UNIVERSITY. — All courses are three hours a week. The Roman numerals refer to the first (I) and second (II) term. — By Professor H. B. FINE: Theory of algebraic functions, I. — By Professor H. D. THOMPSON: Historical readings in infinitesimal geometry, I. — By Professor J. H. JEANS: Electricity and magnetism, I, II; Molecular dynamics and the kinetic theory, I, II. — By Professor G. A. BLISS: Linear differential equations, I; Partial differential equations, II. — By Professor L. P. EISENHART: Differential geometry, I, II. — By Professor W. GILLESPIE: Theory of substitutions,

I; Theory of invariants, II. — By Professor O. VEBLEN: Projective geometry, I, II. — By Professor J. W. YOUNG. Theory of functions of a complex variable, I, II; Theory of numbers, I. — By Professor BLISS or VEBLEN: Theory of functions of a real variable, I, II. — By Dr. J. G. HUN: Analytic projective geometry, I, II. — By Dr. C. R. MACINNES: Elliptic functions, II. — By Dr. R. L. MOORE: Foundations of geometry, II. — By Dr. C. E. STROMQUIST: Calculus of variations, II. — By Dr. E. SWIFT: Theory of capillarity, II.

THE following courses are announced for the summer semester of 1908:

UNIVERSITY OF MUNICH. — By Professor F. LINDEMANN: Integral calculus, five hours; Conformal representation and linear differential equations, four hours; Foundations of geometry, two hours; Seminar, one and one half hours. — By Professor H. v. SEELINGER: Theory of probabilities and method of least squares, two hours. — By Professor A. VOSS: Analytic geometry of space, four hours; Analytic mechanics, II, four hours; Seminar, two hours. — By Professor A. PRINGSHEIM: Selected chapters in the theory of analytic functions, four hours; Elementary theory of ordinary differential equations, three hours. — By Professor K. DOEHLEMANN: Descriptive geometry, II, with exercises, five hours; Synthetic geometry, II, four hours; Imaginary elements of geometry, one hour. — By Professor H. BRUNN: Elements of higher mathematics, four hours. — By Dr. F. HARTOGS: Theory of finite groups and algebraic equations, four hours. — By Dr. O. PERRON: Solid geometry and spherical trigonometry, two hours; Calculus of variations, two hours.

UNIVERSITY OF STRASSBURG. — By Professor Th. REYE: Geometry of position, four hours; Seminar, two hours. — By Professor H. WEBER: Definite integrals and introduction to the theory of functions, four hours; Encyclopedia of elementary mathematics, two hours; Seminar, two hours. — By Professor J. WELLSTEIN: Introduction to the theory of invariants, four hours; Ultraelliptic functions, two hours; Seminar, two hours. — By Professor H. E. TIMERDING: Analytic geometry of space, four hours; with exercises, one hour; Mechanics, four hours. — By Dr. P. EPSTEIN: Introduction to the theory of

numbers, two hours. — By Professor M. SIMON: History of mathematics in ancient times, two hours.

MR. W. W. WALLACE has been appointed professor of mechanics at the University of Liverpool.

MR. D. K. PICKEN, of the University of Glasgow, has been appointed professor of mathematics at Victoria College, Wellington, New Zealand.

DR. R. FUETER, of the University of Marburg, has been appointed associate professor of mathematics at the University of Basel.

DR. E. MEYER has been appointed docent in projective and descriptive geometry at the technical school of Charlottenburg.

DR. F. DALWIGK, of the University of Marburg, has been promoted to an associate professorship of mathematics.

AT Columbia University Professor H. B. MITCHELL has been advanced to a full professorship of mathematics.

AT the University of Chicago, Professors J. W. A. YOUNG and H. E. SLAUGHT have been promoted to associate professorships of mathematics; Professors K. LAVES and F. R. MOULTON have been promoted to associate professorships in astronomy.

MR. W. P. RUSSELL, of Pomona College, Claremont, Calif., has been promoted to an assistant professorship of mathematics.

DR. ROXANNA VIVIAN, of Wellesley College, who has been during the past two years professor of mathematics at the American College for Girls in Constantinople, has been promoted to an associate professorship of mathematics at Wellesley and will resume her work there next fall.

PROFESSOR HEINRICH MASCHKE, of the University of Chicago, died March 1, 1908, at the age of 55 years. He was born in Breslau, Germany, graduated from the University of Göttingen, and had been associated with the University of Chicago since 1892, having been made full professor of mathematics in 1907. He contributed extensively to the theory of linear groups and differential invariants. He had been a member of the AMERICAN MATHEMATICAL SOCIETY since 1893, and was a vice-president of the Society in 1907.

THE following catalogues of second-hand mathematical books have recently appeared: K. F. Koehler's Antiquarium, Leipzig, Kurprinzstrasse 6, catalogue No. 572, 2586 titles in pure and applied mathematics; catalogue No. 573, 546 titles of astronomical, mathematical and physical periodicals. — Buchhandlung Gustav Fock, Leipzig, Schlossgasse 7, catalogue No. 325, 2275 titles in mathematics, physics and astronomy. — W. Heffer and Sons, Cambridge, England, catalogue No. 35, 2825 titles in mathematics and physical sciences. — James Thin, Edinburgh, 54–55 South Bridge, catalogue No. 158, about 1000 titles in mathematics, astronomy, and physical science.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BERTRAND (G.). Sur une fonction déduite de la fonction O de M. Appell. Les rapports avec la fonction σ. (Thèse.) Paris, 1908. 8vo. 36 pp.

CRATHORNE (A. R.). Das räumliche isoperimetrische Problem. (Diss.) Göttingen, 1907. 8vo. 59 pp.

CUTRONI (F.). Alcuni teoremi di geometria negli spazii a n dimensioni. Nota I (Parallelismo e perpendicolarità). Messina, d'Angelo, 1908. 8vo. 10 pp.

DANTSCHER (V. VON). Vorlesungen über die Weierstrasssche Theorie der irrationalen Zahlen. Leipzig, Teubner, 1908. 8vo. 6 + 80 pp. Cloth. M. 3.40

ENCYCLOPÉDIE des sciences mathématiques pures et appliquées. Edition française publiée sous la direction de J. Molk. Tome I; deuxième volume; premier fascicule: Les fonctions rationnelles; exposé, d'après l'article allemand de E. Netto, par R. Le Vavasseur. Paris, Gauthier-Villars, 1908. 8vo. 232 pp.

GURSKI (V.). Ueber den Zusammenhang zwischen den partikulären Lösungen der einzelnen Gebiete bei der hypergeometrischen Differentialgleichung dritter Ordnung mit zwei endlichen singulären Punkten. Berlin, Trenkel, 1907. 8vo. 60 pp. M. 2.00

HANSEN (O.). Ueber die äquiforme Geometrie im Bündel. (Diss.) Kiel, 1907. 8vo. 49 pp.

HEIBERG (J. L.) und ZEUTHEN (G. H.). Eine neue Schrift des Archimedes. Leipzig, Teubner, 1907. 8vo. 2 + 43 pp. M. 1.60

HELLINGER (E.). Die Orthogonalinvarianten quadratischer Formen von unendlichvielen Variabeln. (Diss.) Göttingen, 1907.

HELLWIG (M.). Untersuchung über die Lage der Inzidenzpunkte bei Reflexion und Refraktion an Ebene, Kugel und Kreiszylinder für zwei feste Punkte im Raum (Licht- und Augenpunkt). (Diss.) Rostock, 1906. 8vo. 30 pp.

HÜTTIG (F.). Arithmetische Theorie eines Galoisschen Körpers. (Diss.) Marburg, 1907. 8vo. 40 pp.

KIEPERT (L.). Grundriss der Differential- und Integral-Rechnung. 2ter Teil: Integral-Rechnung. 9te verbesserte und vermehrte Auflage des gleichnamigen Leitfadens von M. Stegemann. Hannover, Helwing, 1908. 8vo. 23 + 738 pp. Cloth. M. 13.50

——. Tabelle der wichtigsten Formeln aus der Integral-Rechnung. [Aus: Grundriss der Integral-Rechnung, 9te Auflage.] Hannover, Helwing, 1908. 8vo. 59 pp. M. 0.50

KNOPP (K.). Grenzwerte von Reihen bei der Annäherung an die Konvergenzgrenze. (Diss.) Berlin, 1907. 8vo. 50 pp.

KÖNIG (R.). Die Oszillationseigenschaften der Eigenfunktionen der Integralgleichung mit definitem Kern und das Jacobische Kriterium der Variationsrechnung. (Diss.) Göttingen, 1907.

LALANDE (A.). Lectures sur la philosophie des sciences (Classes de mathématiques). Nouvelle édition, revue et augmentée conformément aux programmes d'admission à l'Ecole polytechnique et à l'Ecole de Saint-Cyr. Paris, Hachette, 1907. 16mo. 11 + 382 pp. Fr. 3.50

LE VAVASSEUR (R.). See ENCYCLOPÉDIE.

MATISSE (G.). Le principe de la conservation de l'assise et ses applications. Paris, Hermann, 1908. 8vo. 65 pp. Fr. 2.50

MOLK (J.). See ENCYCLOPÉDIE.

MONTVERDE (J.). Primeras teorías de geometría fundamentadas sin le auxilio del postulado de Euclídes. Madrid, 1908. 79 pp. P. 3.00

MORGENSTERN (A.). Beiträge zur numerischen Lösung der Gleichungen 5ten Grades. (Diss.) Halle, 1907. 8vo. 44 pp.

NETTO (E.). See ENCYCLOPÉDIE.

PECK (J. H.). La formule $\rho = ne^{i(\phi+i\psi)}$ interprétée géométriquement dans l'espace de manière à prendre la forme d'un quaternion. Amsterdam, Eisendrath, 1908. 8vo. 24 pp.

SCHWARZLOSE (R.). Ueber eine geometrische Beziehung der Fresnelschen Wellenfläche zum dreiachsigen Ellipsoid. (Diss.) Rostock, 1906. 8vo. 56 pp.

STAROSTZIK (H.). Ueber eine von Steiner gefundenc, noch wenig beachtete Eigenschaft der Leitstrahlen der Kegelschnitte und über Kurven, die mit den einen Kegelschnittt doppelt berührenden Kreisen zusammenhängen. (Diss.). Breslau, 1907. 8vo. 121 pp.

STEGEMANN (M.). See KIEPERT (L.).

STUYVAERT (M.). Cinq études de géométrie analytique. Applications diverses de la théorie des matrices et de l'élimination. 8vo. 230 pp. Fr. 6.00

VOGT (H.). Eléments de mathématiques supérieures. Edition réduite, à l'usage des élèves chimistes et des élèves des écoles industrielles. Paris, Vuibert, 1908. 8vo. 6 + 360 pp.

WEINBAUM (O.). Die Spiegelung einer unendlichen Ebene in einem zu ihr senkrechten elliptischen Zylinder. (Diss.) Rostock, 1906. 8vo. 50 pp.

WEISSE (E.). Anwendung der elliptischen Funktionen auf ein Problem aus der Theorie der Gelenkmechanismen. (Diss.) Rostock, 1907.

WORMS DE ROMILLY (P.). Sur les premiers principes des sciences mathématiques. Paris, Hermann, 1908. 8vo. 55 pp.

ZEUTHEN (G. H.). See HEIBERG (J. L.).

II. ELEMENTARY MATHEMATICS.

ANLEITUNG, praktische, zur Lösung von algebraischen Aufgaben mit Gleichungen. 3te Auflage. Arnsberg, Stahl, 1908. 8vo. 56 pp. M. 0.60

CATANIA (S.). Elementi di aritmetica ed algebra, per le scuole secondarie inferiori. Catania, Giannotta, 1908. 16mo. 8 + 279 pp. L. 2.00

DREYFUS (L.). See FORT (L.).

DRUXES (J.). See HEIS (E.).

FORT (L.) et DREYFUS (L.). Eléments de géométrie conformes aux programmes du 27 juillet 1905 (classes de quatrième *B* et de troisième *A*), avec de nombreux exercices, figures et problèmes. Vol. 2. Paris, Paulin, 1908. 18mo. Pp. 217–418. Fr. 2.50

HEIS (E.). Sammlung von Beispielen und Aufgaben aus der allgemeinen Arithmetik und Algebra zum Gebrauche an höheren Schulen. 112te Auflage. Nach den zeitgemässen Anforderungen des mathematischen Unterrichts neubearbeitet und erweitert von J. Druxes. 1er Teil: Pensum der Unter-, Ober-Tertia und Unter-Sekunda. Köln, Du Mont-Schauberg, 1908. 8vo. 5 + 280 pp. M. 3.00

MOUGIN (E.). Nouvelles tables de logarithmes à l'usage des écoles. Paris, 1907. 4to. Fr. 1.50

SCHMID (A.). See SICKENBERGER (A.).

SICKENBERGER (A.). Leitfaden der elementaren Mathematik. 3er Teil: Stereometrie; Trigonometrie. 5te Auflage. Bearbeitet von A. Schmid. München, Ackermann, 1907. 8vo. 6 + 107 pp. M. 1.35

SIMON (P.). Traité de géométrie élémentaire, à l'usage des élèves de l'enseignement secondaire (premier et second cycles), suivi de compléments, à l'usage des candidats aux écoles du gouvernement. (Ouvrage conforme aux programmes du 31 mai 1902). Géométrie de l'espace. Paris, Berlin, 1908. 8vo. Pp. 451–714. Fr. 2.50

TURNER (G. C.). Graphics applied to arithmetic, mensuration, and statics. London, Macmillan, 1908. 8vo. 398 pp. Cloth. 6 s.

WHITE (W. F.). Scrapbook of elementary mathematics. Chicago, Open Court Pub. Co., 1908. 12mo. 248 pp. Cloth. $1.00

III. APPLIED MATHEMATICS.

BAILLIF (L.). Sur un problème de cinématique navale. Paris, Chapelot, 1907. 8vo. 7 pp.

BOLTZMANN (L.). See BUCHHOLZ (H.).

BUCHHOLZ (H.). Das mechanische Potential nach Vorlesungen von L. Boltzmann bearbeitet, und die Theorie der Figur der Erde, zur Einführung in die höhere Geodäsie (angewandte Mathematik). 1er Teil. Leipzig, Barth, 1908. 8vo. 16 + 470 pp. Cloth. M. 16.00

JORDAN (H.). Zur Kenntnis der elastischen Nachwirkung bei kleinen gleichförmigen Biegungen homogener Metallstäbe. (Diss.) Göttingen, 1907. 8vo. 86 pp.

HIRSCHFELD (C. F.). Engineering thermodynamics. London, Spon, 1908. 12mo. 2 s.

HOSKINS (L. M.). Theoretical mechanics: an elementary textbook. 3d edition. Stanford University, Cal., 1908. 8vo. 11 + 456 pp. Cloth. $3.00

LEA (F. C.). Hydraulics. London, Arnold, 1908. 8vo. 548 pp. Cloth. 18 s.

PEABODY (C. H.). Computations for marine engines; prepared for the use of students at the Massachusetts Institute of Technology. Boston, Peabody, 1908. 8vo. 108 pp.

POINCARE (H.). La théorie de Maxwell et les oscillations hertziennes. La télégraphie sans fil. 3e édition. Paris, Gauthier-Villars, 1907. 8vo. 114 pp. Fr. 2.00

WHITTAKER. Arithmetic of electrical engineering. For technical students and engineers. London, Whittaker, 1908. 8vo. 168 pp. 1 s.

WIESSNER (V.). Die mechanische Energie, das Prinzip der Mechanik. Dresden, Lincke, 1908. 8vo. 7 + 253 pp. M. 4.00